

# Approximation by $q$ -Szász operators

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## Abstract

This paper deals with approximating properties of the newly defined  $q$ -generalization of the Szász operators in the case  $q > 1$ . Quantitative estimates of the convergence in the polynomial weighted spaces and the Voronovskaja's theorem are given. In particular, it is proved that the rate of approximation by the  $q$ -Szász operators ( $q > 1$ ) is of order  $q^{-n}$  versus  $1/n$  for the classical Szász–Mirakjan operators.

**Keywords:** Positive linear operators, Szász–Mirakjan operators, Voronovskaja-type asymptotic formula, weighted space, direct results, Korovkin theorem

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## 1 Introduction

The approximation of functions by using linear positive operators introduced via  $q$ -Calculus is currently under intensive research. The pioneer work has been made by A. Lupas [2] and G. M. Phillips [3] who proposed generalizations of Bernstein polynomials based on the  $q$ -integers. The  $q$ -Bernstein polynomials quickly gained the popularity, see [4]–[9]. Other important classes of discrete operators have been investigated by using  $q$ -Calculus in the case  $0 < q < 1$ , for example  $q$ -Meyer–König operators [11], [12], [13],  $q$ -Bleimann, Butzer and Hahn operators [14], [15], [16],  $q$ -Szász–Mirakjan operators [17], [18], [20], [19],  $q$ -Baskakov operators [21].

In the present paper, we introduce a  $q$ -generalization of the Szász operators in the case  $q > 1$ . Notice that different  $q$ -generalizations of Szász–Mirakjan operators were introduced and studied by A. Aral and V. Gupta [17], [18], by C. Radu [20] and by N. I. Mahmudov [19] in the case  $0 < q < 1$ . Since we define  $q$ -Szász operators for  $q > 1$ , the rate of approximation by the  $q$ -Szász operators ( $q > 1$ ) is of order  $q^{-n}$ , which is essentially better than  $1/n$  (rate of approximation for the classical Szász–Mirakjan operators). Thus our  $q$ -Szász operators have better approximation properties than the classical Szász–Mirakjan operators and the other  $q$ -Szász–Mirakjan operators.

The paper is organized as follows. In Section 2, we give standard notations that will be used throughout the paper, introduce  $q$ -Szász operators and evaluate the moments of  $M_{n,q}$ . In Section 3 we study convergence properties of the  $q$ -Szász operators in the polynomial weighted spaces. In Section 4, we give the quantitative Voronovskaja-type asymptotic formula.

## 2 Construction of $M_{n,q}$ and estimation of moments

Throughout the paper we employ the standard notations of  $q$ -calculus, see [25], [24].

$q$ -integer and  $q$ -factorial are defined by

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q} & \text{if } q \in R^+ \setminus \{1\}, \\ n & \text{if } q = 1 \end{cases} \quad \text{for } n \in N \quad \text{and} \quad [0] = 0,$$

$$[n]_q! := [1]_q [2]_q \dots [n]_q \quad \text{for } n \in N \quad \text{and} \quad [0]! = 1.$$

For integers  $0 \leq k \leq n$   $q$ -binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

The  $q$ -derivative of a function  $f(x)$ , denoted by  $D_q f$ , is defined by

$$(D_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad (D_q f)(0) := \lim_{x \rightarrow 0} (D_q f)(x).$$

The formula for the  $q$ -derivative of a product and quotient are

$$D_q(u(x)v(x)) = D_q(u(x))v(x) + u(qx)D_q(v(x)). \quad (1)$$

Also, it is known that

$$D_q x^n = [n] x^{n-1}, \quad D_q E(ax) = aE(qax). \quad (2)$$

If  $|q| > 1$ , or  $0 < |q| < 1$  and  $|z| < \frac{1}{1-q}$ , the  $q$ -exponential function  $e_q(x)$  was defined by Jackson

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}. \quad (3)$$

If  $|q| > 1$ ,  $e_q(z)$  is an entire function and

$$e_q(z) = \prod_{j=0}^{\infty} \left( 1 + (q-1) \frac{z}{q^{j+1}} \right), \quad |q| > 1. \quad (4)$$

There is another  $q$ -exponential function which is entire when  $0 < |q| < 1$  and which converges when  $|z| < \frac{1}{1-q}$  if  $|q| > 1$ . To obtain it we must invert the base in (3), i.e.  $q \rightarrow \frac{1}{q}$ :

$$E_q(z) := e_{1/q}(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{[k]_q!}.$$

We immediately obtain from (4) that

$$E_q(z) = \prod_{j=0}^{\infty} (1 + (1-q) z q^j), \quad 0 < |q| < 1.$$

The  $q$ -difference equations corresponding to  $e_q(z)$  and  $E_q(z)$  are

$$D_q e_q(az) = a e_q(qz), \quad D_q E_q(az) = a E_q(qaz),$$

$$D_{1/q} e_q(z) = D_{1/q} E_{1/q}(z) = E_{1/q}(q^{-1}z) = e_q(q^{-1}z), \quad q \neq 0.$$

Let  $C_p$  is the set of all real valued functions  $f$ , continuous on  $[0, \infty)$  and such that  $w_p f$  is uniformly continuous and bounded on  $[0, \infty)$  endowed with the norm

$$\|f\|_p := \sup_{x \in [0, \infty)} w_p(x) |f(x)|.$$

Here

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1}, \quad \text{if } p \in N.$$

The corresponding Lipschitz classes are given for  $0 < \alpha \leq 2$  by

$$\Delta_h^2 f(x) := f(x+2h) - 2f(x+h) + f(x),$$

$$\omega_p^2(f; \delta) := \sup_{0 < h \leq \delta} \|\Delta_h^2 f\|_p, \quad Lip_p^2 \alpha := \{f \in C_p : \omega_p^2(f; \delta) = O(\delta^\alpha), \quad \delta \rightarrow 0^+\}.$$

Now we introduce the  $q$ -parametric Szász operator.

**Definition 1** Let  $q > 1$  and  $n \in \mathbb{N}$ . For  $f : [0, \infty) \rightarrow \mathbb{R}$  we define the Szász operator based on the  $q$ -integers

$$M_{n,q}(f; x) := \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right) \frac{1}{q^{k(k-1)/2}} \frac{[n]^k x^k}{[k]!} e_q(-[n]q^{-k}x). \quad (5)$$

Similarly as a classical Szász operator  $S_n$ , the operator  $M_{n,q}$  is linear and positive. Furthermore, in the case of  $q \rightarrow 1^+$  we obtain classical Szász–Mirakjan operators.

Moments  $M_{n,q}(t^m; x)$  are of particular importance in the theory of approximation by positive operators. From (5) we easily derive the following recurrence formula and explicit formulas for moments  $M_{n,q}(t^m; x)$ ,  $m = 0, 1, 2, 3, 4$ .

**Lemma 2** Let  $q > 1$ . The following recurrence formula holds

$$M_{n,q}(t^{m+1}; x) = \sum_{j=0}^m \binom{m}{j} \frac{xq^j}{[n]^{m-j}} M_{n,q}(t^j; q^{-1}x). \quad (6)$$

**Proof.** The recurrence formula (6) easily follows from the definition of  $M_{n,q}$  and  $q[k] + 1 = [k+1]$ .

$$\begin{aligned} M_{n,q}(t^{m+1}; x) &= \sum_{k=0}^{\infty} \frac{[k]^{m+1}}{[n]^{m+1}} \frac{1}{q^{k(k-1)/2}} \frac{[n]^k x^k}{[k]!} e_q(-[n]q^{-k}x) \\ &= \sum_{k=1}^{\infty} \frac{[k]^m}{[n]^m} \frac{1}{q^{k(k-1)/2}} \frac{[n]^{k-1} x^k}{[k-1]!} e_q(-[n]q^{-k}x) \\ &= \sum_{k=0}^{\infty} \frac{(q[k] + 1)^m}{[n]^m} \frac{1}{q^{k(k+1)/2}} \frac{[n]^k x^{k+1}}{[k]!} e_q(-[n]q^{-k}q^{-1}x) \\ &= \sum_{k=0}^{\infty} \frac{1}{[n]^m} \sum_{j=0}^m \binom{m}{j} q^j [k]^j \frac{1}{q^{k(k+1)/2}} \frac{[n]^k x^{k+1}}{[k]!} e_q(-[n]q^{-k}q^{-1}x) \\ &= \sum_{j=0}^m \binom{m}{j} \frac{xq^j}{[n]^{m-j}} \sum_{k=0}^{\infty} \frac{[k]^j}{[n]^j} \frac{1}{q^{k(k-1)/2}} \frac{[n]^k x^k}{[k]!} e_q(-[n]q^{-k}q^{-1}x) \\ &= \sum_{j=0}^m \binom{m}{j} \frac{xq^j}{[n]^{m-j}} M_{n,q}(t^j; q^{-1}x). \end{aligned}$$

■

**Lemma 3** The following identities hold for all  $q > 1$ ,  $x \in [0, \infty)$ ,  $n \in \mathbb{N}$ , and  $k \geq 0$ :

$$xD_q s_{nk}(q; x) = [n] \left( \frac{[k]}{[n]} - x \right) s_{nk}(q; x),$$

$$M_{n,q}(t^{m+1}; x) = \frac{x}{[n]} D_q M_{n,q}(t^m; x) + x M_{n,q}(t^m; x). \quad (7)$$

**Proof.** The first identity follows from the following simple calculations

$$\begin{aligned} xD_q s_{nk}(q; x) &= [k]_q \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q(-[n]_q q^{-k} x) - x q^{-k} [n]_q \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k q^k x^k}{[k]_q!} e_q(-[n]_q q^{-k} x) \\ &= [k]_q s_{nk}(q; x) - x [n]_q s_{nk}(q; x) = [n] \left( \frac{[k]}{[n]} - x \right) s_{nk}(q; x). \end{aligned}$$

The second one follows from the first.

$$\begin{aligned} xD_q M_{n,q}(t^m; x) &= [n] \sum_{k=0}^{\infty} \left( \frac{[k]}{[n]} \right)^m \left( \frac{[k]}{[n]} - x \right) s_{nk}(q; x) \\ &= [n] \sum_{k=0}^{\infty} \left( \frac{[k]}{[n]} \right)^{m+1} s_{nk}(q; x) - [n] x \sum_{k=0}^{\infty} \left( \frac{[k]}{[n]} \right)^m s_{nk}(q; x) \\ &= [n] M_{n,q}(t^{m+1}; x) - [n] x M_{n,q}(t^m; x). \end{aligned}$$

■

**Lemma 4** Let  $q > 1$ . We have

$$\begin{aligned} M_{n,q}(1; x) &= 1, \quad M_{n,q}(t; x) = x, \quad M_{n,q}(t^2; x) = x^2 + \frac{1}{[n]}x, \\ M_{n,q}(t^3; x) &= x^3 + \frac{2+q}{[n]}x^2 + \frac{1}{[n]^2}x, \\ M_{n,q}(t^4; x) &= x^4 + (3+2q+q^2) \frac{x^3}{[n]} + (3+3q+q^2) \frac{x^2}{[n]^2} + \frac{1}{[n]^3}x. \end{aligned}$$

**Proof.** For a fixed  $x \in R_+$ , by the  $q$ -Taylor theorem [25], we obtain

$$\varphi_n(t) = \sum_{k=0}^{\infty} \frac{(t-x)_{1/q}^k}{[k]_{1/q}!} D_{1/q}^k \varphi_n(x).$$

Choosing  $t = 0$  and taking into account

$$(-x)_{1/q}^k = (-1)^k x^k q^{-k(k-1)/2}, \quad D_{1/q}^k e_q(-[n]_q x) = (-1)^k q^{-k(k-1)/2} [n]_q^k e_q(-[n]_q q^{-k} x)$$

we get for  $\varphi_n(x) = e_q(-[n]x)$  that

$$\begin{aligned} 1 &= \varphi_n(0) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{q^{k(k-1)/2} [k]_{1/q}!} D_{1/q}^k \varphi_n(x) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{[k]_q!} (-1)^k q^{-k(k-1)/2} [n]_q^k e_q(-[n]_q q^{-k} x) \\ &= \sum_{k=0}^{\infty} \frac{[n]_q^k x^k}{[k]_q! q^{k(k-1)/2}} e_q(-[n]_q q^{-k} x). \end{aligned}$$

In other words  $M_{n,q}(1; x) = 1$ .

Calculation of  $M_{n,q}(t^i; x)$ ,  $i = 1, 2, 3, 4$ , based on the recurrence formula (7) (or (6)). We only calculate  $M_{n,q}(t^3; x)$  and  $M_{n,q}(t^4; x)$ :

$$\begin{aligned} M_{n,q}(t^3; x) &= \frac{x}{[n]} D_q M_{n,q}(t^2; x) + x M_{n,q}(t^2; x) \\ &= \frac{x}{[n]} \left( [2]x + \frac{1}{[n]} \right) + x \left( x^2 + \frac{1}{[n]}x \right) \\ &= \frac{1}{[n]^2}x + \frac{2+q}{[n]}x^2 + x^3. \end{aligned}$$

$$\begin{aligned}
M_{n,q}(t^4; x) &= \frac{x}{[n]} D_q M_{n,q}(t^3; x) + x M_{n,q}(t^3; x) \\
&= \frac{x}{[n]} \left( \frac{1}{[n]^2} + \frac{2+q}{[n]} [2]x + [3]x^2 \right) + x \left( \frac{1}{[n]^2}x + \frac{2+q}{[n]}x^2 + x^3 \right) \\
&= \frac{1}{[n]^3}x + (3+3q+q^2) \frac{x^2}{[n]^2} + (3+2q+q^2) \frac{x^3}{[n]} + x^4.
\end{aligned}$$

■

**Lemma 5** Assume that  $q > 1$ . For every  $x \in [0, \infty)$  there hold

$$M_{n,q}((t-x)^2; x) = \frac{x}{[n]}, \quad (8)$$

$$M_{n,q}((t-x)^3; x) = \frac{1}{[n]^2}x + (q-1) \frac{x^2}{[n]}, \quad (9)$$

$$M_{n,q}((t-x)^4; x) = \frac{1}{[n]^3}x + (q^2+3q-1) \frac{x^2}{[n]^2} + (q-1)^2 \frac{x^3}{[n]}. \quad (10)$$

**Proof.** First of all we give an explicit formula for  $M_{n,q}((t-x)^4; x)$ .

$$\begin{aligned}
M_{n,q}((t-x)^3; x) &= M_{n,q}(t^3; x) - 3x M_{n,q}(t^2; x) + 3x^2 M_{n,q}(t; x) - x^3 \\
&= x^3 + \frac{2+q}{[n]}x^2 + \frac{1}{[n]^2}x - 3x \left( x^2 + \frac{x}{[n]} \right) + 3x^3 - x^3 \\
&= \frac{1}{[n]^2}x + (q-1) \frac{x^2}{[n]}.
\end{aligned}$$

$$\begin{aligned}
M_{n,q}((t-x)^4; x) &= M_{n,q}(t^4; x) - 4x M_{n,q}(t^3; x) + 6x^2 M_{n,q}(t^2; x) - 4x^3 M_{n,q}(t; x) + x^4 \\
&= \frac{1}{[n]^3}x + (3+3q+q^2) \frac{x^2}{[n]^2} + (3+2q+q^2) \frac{x^3}{[n]} + x^4 \\
&\quad - 4x \left( \frac{1}{[n]^2}x + \frac{2+q}{[n]}x^2 + x^3 \right) + 6x^2 \left( x^2 + \frac{x}{[n]} \right) - 4x^4 + x^4 \\
&= \frac{1}{[n]^3}x + (-1+3q+q^2) \frac{x^2}{[n]^2} + (q-1)^2 \frac{x^3}{[n]}.
\end{aligned}$$

■

Now we prove explicit formula for the moments  $M_{n,q}(t^m; x)$ , which a  $q$ -analogue of a result of Becker, see [22] Lemma 3.

**Lemma 6** For  $q > 1$ ,  $m \in \mathbb{N}$  there holds

$$M_{n,q}(t^m; x) = \sum_{j=1}^m \mathbb{S}_q(m, j) \frac{x^j}{[n]^{m-j}}, \quad (11)$$

where

$$\begin{aligned}
\mathbb{S}_q(m+1, j) &= [j] \mathbb{S}_q(m, j) + \mathbb{S}_q(m, j-1), \quad m \geq 0, \quad j \geq 1, \\
\mathbb{S}_q(0, 0) &= 1, \quad \mathbb{S}_q(m, 0) = 0, \quad m > 0, \quad \mathbb{S}_q(m, j) = 0, \quad m < j.
\end{aligned} \quad (12)$$

In particular  $M_{n,q}(t^m; x)$  is a polynomial of degree  $m$  without a constant term.

**Proof.** Because of  $M_{n,q}(t; x) = x$ ,  $M_{n,q}(t^2; x) = x^2 + \frac{x}{[n]}$ , the representation (11) holds true for  $m = 1, 2$  with  $\mathbb{S}_q(2, 1) = 1$ ,  $\mathbb{S}_q(1, 1) = 1$ .

Now assume (11) to be valued for  $m$  then by Lemma 3 we have

$$\begin{aligned}
M_{n,q}(t^{m+1}; x) &= \frac{x}{[n]} D_q M_{n,q}(t^m; x) + x M_{n,q}(t^m; x) \\
&= \frac{x}{[n]} \sum_{j=1}^m [j] \mathbb{S}_q(m, j) \frac{x^{j-1}}{[n]^{m-j}} + x \sum_{j=1}^m \mathbb{S}_q(m, j) \frac{x^j}{[n]^{m-j}} \\
&= \sum_{j=1}^m [j] \mathbb{S}_q(m, j) \frac{x^j}{[n]^{m-j+1}} + \sum_{j=1}^m \mathbb{S}_q(m, j) \frac{x^{j+1}}{[n]^{m-j}} \\
&= \frac{x}{[n]^m} \mathbb{S}_q(m, 1) + x^{m+1} \mathbb{S}_q(m, m) \\
&\quad + \sum_{j=2}^m ([j] \mathbb{S}_q(m, j) + \mathbb{S}_q(m, j-1)) \frac{x^j}{[n]^{m-j+1}}.
\end{aligned}$$

■

**Remark 7** For  $q = 1$  the formulae (12) become recurrence formulas satisfied by Stirling numbers of the second type.

### 3 $M_{n,q}$ in polynomial weighted spaces

**Lemma 8** Let  $p \in N \cup \{0\}$  and  $q \in (1, \infty)$  be fixed. Then there exists a positive constant  $K_1(q, p)$  such that

$$\|M_{n,q}(1/w_p; x)\|_p \leq K_1(q, p), \quad n \in N. \quad (13)$$

Moreover for every  $f \in C_p$  we have

$$\|M_{n,q}(f)\|_p \leq K_1(q, p) \|f\|_p, \quad n \in N. \quad (14)$$

Thus  $M_{n,q}$  is a linear positive operator from  $C_p$  into  $C_p$  for any  $p \in N \cup \{0\}$ .

**Proof.** The inequality (13) is obvious for  $p = 0$ . Let  $p \geq 1$ . Then by (11) we have

$$w_p(x) M_{n,q}(1/w_p; x) = w_p(x) + w_p(x) \sum_{j=1}^p \mathbb{S}_q(p, j) \frac{x^j}{[n]^{p-j}} \leq K_1(q, p),$$

$K_1(q, p)$  is a positive constant depending on  $p$  and  $q$ . From this follows (13). On the other hand

$$\|M_{n,q}(f)\|_p \leq \|f\|_p \|M_{n,q}(1/w_p)\|_p$$

for every  $f \in C_p$ . By applying (13), we obtain (14). ■

**Lemma 9** Let  $p \in N \cup \{0\}$  and  $q \in (1, \infty)$  be fixed. Then there exists a positive constant  $K_2(q, p)$  such that

$$\left\| M_{n,q} \left( \frac{(t - \cdot)^2}{w_p(t)}; \cdot \right) \right\|_p \leq \frac{K_2(q, p)}{[n]}, \quad n \in N. \quad (15)$$

**Proof.** The formula (8) imply (15) for  $p = 0$ . We have

$$M_{n,q} \left( \frac{(t - x)^2}{w_p(t)}; x \right) = M_{n,q} \left( (t - x)^2; x \right) + M_{n,q} \left( (t - x)^2 t^p; x \right),$$

for  $p, n \in N$ . If  $p = 1$  then we get

$$\begin{aligned} M_{n,q} \left( (t-x)^2 (1+t); x \right) &= M_{n,q} \left( (t-x)^2; x \right) + M_{n,q} \left( (t-x)^2 t; x \right) \\ &= M_{n,q} \left( (t-x)^3; x \right) + (1+x) M_{n,q} \left( (t-x)^2; x \right), \end{aligned}$$

which by Lemma 5 yields (15) for  $p = 1$ .

Let  $p \geq 2$ . By applying (11), we get

$$\begin{aligned} &w_p(x) M_{n,q} \left( (t-x)^2 t^p; x \right) \\ &= w_p(x) \left( M_{n,q} \left( t^{p+2}; x \right) - 2x M_{n,q} \left( t^{p+1}; x \right) + x^2 M_{n,q} \left( t^p; x \right) \right) \\ &= w_p(x) \left( x^{p+2} + \sum_{j=1}^{p+1} \mathbb{S}_q(p+2, j) \frac{x^j}{[n]^{p+2-j}} - 2x^{p+2} - 2 \sum_{j=1}^p \mathbb{S}_q(p+1, j) \frac{x^{j+1}}{[n]^{p+1-j}} + x^{p+2} + \sum_{j=1}^{p-1} \mathbb{S}_q(p, j) \frac{x^{j+2}}{[n]^{p-j}} \right) \\ &= w_p(x) \left( \sum_{j=2}^p (\mathbb{S}_q(p+2, j) - 2\mathbb{S}_q(p+1, j) + \mathbb{S}_q(p, j)) \frac{x^{j+1}}{[n]^{p+1-j}} \right. \\ &\quad \left. + \mathbb{S}_q(p+2, 1) \frac{x}{[n]^{p+1}} + (\mathbb{S}_q(p+2, 2) - 2\mathbb{S}_q(p+2, 1)) \frac{x^2}{[n]^p} \right) \\ &= w_p(x) \frac{x}{[n]} \mathcal{P}_p(q; x), \end{aligned}$$

where  $\mathcal{P}_p(q; x)$  is a polynomial of degree  $p$ . Therefore one has

$$w_p(x) M_{n,q} \left( (t-x)^2 t^p; x \right) \leq K_2(q, p) \frac{x}{[n]}.$$

■

Our first main result in this section is a local approximation property of  $M_{n,q}$  stated below.

**Theorem 10** *There exists an absolute constant  $C > 0$  such that*

$$w_p(x) |M_{n,q}(g; x) - g(x)| \leq K_3(q, p) \|g''\| \frac{x}{[n]},$$

where  $g \in C_p^2$ ,  $q > 1$  and  $x \in [0, \infty)$ .

**Proof.** Using the Taylor formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t \int_x^s g''(u) du ds, \quad g \in C_p^2,$$

we obtain that

$$\begin{aligned} w_p(x) |M_{n,q}(g; x) - g(x)| &= w_p(x) \left| M_{n,q} \left( \int_x^t \int_x^s g''(u) du ds; x \right) \right| \\ &\leq w_p(x) M_{n,q} \left( \left| \int_x^t \int_x^s g''(u) du ds \right|; x \right) \\ &\leq w_p(x) M_{n,q} \left( \|g''\|_p \left| \int_x^t \int_x^s (1+u^m) du ds \right|; x \right) \\ &\leq w_p(x) \frac{1}{2} \|g''\|_p M_{n,q} \left( (t-x)^2 (1/w_p(x) + 1/w_p(t)); x \right) \\ &\leq \frac{1}{2} \|g''\|_p \left( M_{n,q} \left( (t-x)^2; x \right) + w_p(x) M_{n,q} \left( (t-x)^2 w_p(t); x \right) \right) \\ &\leq K_3(q, x) \|g''\|_p \frac{x}{[n]}. \end{aligned}$$

■

Now we consider the modified Steklov means

$$f_h(x) := \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x+s+t) - f(x+2(s+t))] ds dt.$$

$f_h(x)$  has the following properties:

$$f(x) - f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \Delta_{s+t}^2 f(x) ds dt, \quad f_h''(x) = h^{-2} \left( 8\Delta_{\frac{h}{2}}^2 f(x) - \Delta_h^2 f(x) \right)$$

and therefore

$$\|f - f_h\|_p \leq \omega_p^2(f; h), \quad \|f_h''\|_p \leq \frac{1}{9h^2} \omega_p^2(f; h).$$

We have the following direct approximation theorem:

**Theorem 11** *For every  $p \in \mathbb{N} \cup \{0\}$ ,  $f \in C_p$  and  $x \in [0, \infty)$ ,  $q > 1$ , we have*

$$w_p(x) |M_{n,q}(f; x) - f(x)| \leq M_p \omega_p^2 \left( f; \sqrt{\frac{x}{[n]}} \right) = M_p \omega_p^2 \left( f; \sqrt{\frac{(q-1)x}{(q^n - 1)}} \right).$$

Particularly, if  $Lip_p^2 \alpha$  for some  $\alpha \in (0, 2]$ , then

$$w_p(x) |M_{n,q}(f; x) - f(x)| \leq M_p \left( \frac{x}{[n]} \right)^{\frac{\alpha}{2}}$$

**Proof.** For  $f \in C_p$  and  $h > 0$

$$|M_{n,q}(f; x) - f(x)| \leq |M_{n,q}((f - f_h); x) - (f - f_h)(x)| + |M_{n,q}(f_h; x) - f_h(x)|$$

and therefore

$$\begin{aligned} w_p(x) |M_{n,q}(f; x) - f(x)| &\leq \|f - f_h\|_p \left( w_p(x) M_{n,q} \left( \frac{1}{w_p(t)}; x \right) + 1 \right) \\ &\quad + K_3(q, p) \|f_h''\|_p \frac{x}{[n]}. \end{aligned}$$

Since  $w_p(x) M_{n,q} \left( \frac{1}{w_p(t)}; x \right) \leq K_1(q, p)$ , we get that

$$w_p(x) |M_{n,q}(f; x) - f(x)| \leq M(q, p) \omega_p^2(f; h) \left[ 1 + \frac{x}{[n] h^2} \right]$$

Thus, choosing  $h = \sqrt{\frac{x}{[n]}}$ , the proof is completed. ■

**Corollary 12** *If  $p \in \mathbb{N} \cup \{0\}$ ,  $f \in C_p$ ,  $q > 1$  and  $x \in [0, \infty)$ , then*

$$\lim_{n \rightarrow \infty} M_{n,q}(f; x) = f(x).$$

*This convergcnce is uniform on every  $[a, b]$ ,  $0 \leq a < b$ .*

**Remark 13** *Theorem 11 shows the rate of approximation by the  $q$ -Szász operators ( $q > 1$ ) is of order  $q^{-n}$  versus  $1/n$  for the classical Szász-Mirakjan operators.*



## 4 Convergence of $q$ -Szász operators

An interesting problem is to determine the class of all continuous functions  $f$  such that  $M_{n,q}(f)$  converges to  $f$  uniformly on the whole interval  $[0, \infty)$  as  $n \rightarrow \infty$ . This problem was investigated by Totik [27, Theorem 1] and de la Cal [23, Theorem 1]. The following result is a  $q$ -analogue of Theorem 1 [23].

**Theorem 14** *Assume that  $f : [0, \infty) \rightarrow \mathbb{R}$  is bounded or uniformly continuous. Let*

$$f^*(z) = f(z^2), \quad z \in [0, \infty).$$

*We have, for all  $t > 0$  and  $x \geq 0$ ,*

$$|M_{n,q}(f; x) - f(x)| \leq 2\omega\left(f^*; \sqrt{\frac{1}{[n]}}\right). \quad (16)$$

*Therefore,  $M_{n,q}(f; x)$  converges to  $f$  uniformly on  $[0, \infty)$  as  $n \rightarrow \infty$ , whenever  $f^*$  is uniformly continuous.*

**Proof.** By the definition of  $f^*$  we have

$$M_{n,q}(f; x) = M_{n,q}(f^*(\sqrt{\cdot}); x).$$

Thus we can write

$$\begin{aligned} |M_{n,q}(f; x) - f(x)| &= |M_{n,q}(f^*(\sqrt{\cdot}); x) - f^*(\sqrt{x})| \\ &= \left| \sum_{k=0}^{\infty} \left( f^*\left(\sqrt{\frac{[k]}{[n]}}\right) - f^*(\sqrt{x}) \right) s_{n,k}(q; x) \right| \\ &\leq \sum_{k=0}^{\infty} \left| \left( f^*\left(\sqrt{\frac{[k]}{[n]}}\right) - f^*(\sqrt{x}) \right) \right| s_{n,k}(q; x) \\ &\leq \sum_{k=0}^{\infty} \omega\left(f^*; \left| \sqrt{\frac{[k]}{[n]}} - \sqrt{x} \right| \right) s_{n,k}(q; x) \\ &\leq \sum_{k=0}^{\infty} \omega\left(f^*; \frac{\left| \sqrt{\frac{[k]}{[n]}} - \sqrt{x} \right|}{M_{n,q}(|\sqrt{\cdot} - \sqrt{x}|; x)} M_{n,q}(|\sqrt{\cdot} - \sqrt{x}|; x) \right) s_{n,k}(q; x). \end{aligned}$$

Finally, from the inequality

$$\omega(f^*; \alpha\delta) \leq (1 + \alpha)\omega(f^*; \delta), \quad \alpha, \delta \geq 0,$$

we obtain

$$\begin{aligned} |M_{n,q}(f; x) - f(x)| &\leq \omega(f^*; M_{n,q}(|\sqrt{\cdot} - \sqrt{x}|; x)) \sum_{k=0}^{\infty} \left( 1 + \frac{\left| \sqrt{\frac{[k]}{[n]}} - \sqrt{x} \right|}{M_{n,q}(|\sqrt{\cdot} - \sqrt{x}|; x)} \right) s_{n,k}(q; x) \\ &= 2\omega(f^*; M_{n,q}(|\sqrt{\cdot} - \sqrt{x}|; x)). \end{aligned}$$

In order to complete the proof we need to show that we have for all  $t > 0$  and  $x > 0$ ,

$$M_{n,q}(|\sqrt{\cdot} - \sqrt{x}|; x) \leq \sqrt{\frac{1}{[n]}}.$$

Indeed we obtain from the Cauchy-Schwarz inequality

$$\begin{aligned}
M_{n,q}(|\sqrt{\cdot} - \sqrt{x}|; x) &= \sum_{k=0}^{\infty} \left| \sqrt{\frac{[k]}{[n]}} - \sqrt{x} \right| s_{n,k}(q; x) \\
&= \sum_{k=0}^{\infty} \frac{\left| \frac{[k]}{[n]} - x \right|}{\sqrt{\frac{[k]}{[n]} + \sqrt{x}}} s_{n,k}(q; x) \leq \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \left| \frac{[k]}{[n]} - x \right| s_{n,k}(q; x) \\
&\leq \frac{1}{\sqrt{x}} \sqrt{\sum_{k=0}^{\infty} \left| \frac{[k]}{[n]} - x \right|^2 s_{n,k}(q; x)} = \frac{1}{\sqrt{x}} \sqrt{M_{n,q}((\cdot - x)^2; x)} \\
&= \frac{1}{\sqrt{x}} \sqrt{\frac{1}{[n]} x} = \sqrt{\frac{1}{[n]}}
\end{aligned}$$

showing (16), and completing the proof. ■

Next we prove Voronovskaja type result for  $q$ -Szász-Mirakjan operators.

**Theorem 15** Assume that  $q \in (1, \infty)$ . For any  $f \in C_p^2$  the following equality holds

$$\lim_{n \rightarrow \infty} [n] (M_{n,q}(f; x) - f(x)) = \frac{1}{2} f''(x) x$$

for every  $x \in [0, \infty)$ .

**Proof.** Let  $x \in [0, \infty)$  be fixed. By the Taylor formula we may write

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + r(t; x)(t - x)^2, \quad (17)$$

where  $r(t; x)$  is the Peano form of the remainder,  $r(\cdot; x) \in C_p$  and  $\lim_{t \rightarrow x} r(t; x) = 0$ . Applying  $M_{n,q}$  to (17) we obtain

$$\begin{aligned}
[n] (M_{n,q}(f; x) - f(x)) &= f'(x) [n] M_{n,q}(t - x; x) \\
&\quad + \frac{1}{2} f''(x) [n] M_{n,q}((t - x)^2; x) + [n] M_{n,q}(r(t; x)(t - x)^2; x).
\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$M_{n,q}(r(t; x)(t - x)^2; x) \leq \sqrt{M_{n,q}(r^2(t; x); x)} \sqrt{M_{n,q}((t - x)^4; x)}. \quad (18)$$

Observe that  $r^2(x; x) = 0$ . Then it follows from Corollary 12 that

$$\lim_{n \rightarrow \infty} M_{n,q}(r^2(t; x); x) = r^2(x; x) = 0. \quad (19)$$

Now from (18), (19) and Lemma 5 we get immediately

$$\lim_{n \rightarrow \infty} [n] M_{n,q}(r(t; x)(t - x)^2; x) = 0.$$

The proof is completed. ■

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